EXISTENCE OF WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS TO SOME CLASSES OF NONAUTONOMOUS PARTIAL EVOLUTION EQUATIONS

TOKA DIAGANA

Abstract. In this paper, under Acquistpace-Terreni conditions, we make extensive use of interpolation spaces and exponential dichotomy techniques to obtain the existence of weighted pseudo almost periodic solutions to some classes of nonautonomous partial evolution equations. Applications include the existence of weighted pseudo almost periodic solutions to a nonautonomous heat equation with gradient coefficients.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $0 < \alpha < \beta < 1$. The main impetus of the present work is a recent paper by the author [21] in which, the existence of weighted pseudo almost periodic solutions to the classes of autonomous partial equations

$$\frac{d}{dt}\bigg[u(t) + f(t, Bu(t))\bigg] = Au(t) + g(t, Cu(t)), \quad t \in \mathbb{R} \tag{1.1}$$

where A is a sectorial operator, B and C are closed linear operators, and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are pseudo almost periodic function in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$. For that, the author assumed that the analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic, equivalently, $\sigma(A) \cap i\mathbb{R} = \emptyset$, and made extensive use of interpolation spaces to obtain the existence of weighted pseudo almost periodic solutions to Eq. (1.1).

In this paper, we consider a more general setting and use slightly different techniques to study the existence of weighted pseudo almost periodic solutions to the the class of abstract nonautonomous differential equations

$$\frac{d}{dt}\bigg[u(t) + f(t, B(t)u(t))\bigg] = A(t)u(t) + g(t, C(t)u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the well-known Acquistapace-Terreni conditions, $B(t), C(t)$ $(t \in \mathbb{R})$ are families of (possibly unbounded) linear operators, and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta^t$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are weighted pseudo almost periodic in $t \in \mathbb{R}$ uniformly in the second variable. Under the previous assumptions, it is well-known that there exists an evolution family $\mathcal{U} = \{U(t, s)\}_{t\geq s}$ associated with the family of operators $A(t)$ $(t \in \mathbb{R})$. Assuming that the evolution family $\mathcal{U} = U(t, s)$ is exponentially dichotomic (hyperbolic) and

1991 *Mathematics Subject Classification.* 34G20; 34B05; 42A75; 47D06; 35L90.

Key words and phrases. evolution family; exponential dichotomy; Acquistapace and Terreni conditions; almost periodic; weighted pseudo almost periodic; evolution equation; nonautonomous equation.

under some additional assumptions it will be shown that Eq. (1.2) has a unique weighted pseudo almost periodic solution.

The main result of this paper (Theorem 3.6) generalizes most of the known results on (weighted) pseudo almost periodic solutions to autonomous and nonautonomous differential equations, especially those in [6, 10, 13, 14, 15, 16, 17, 18, 19, 20, 22, 30, 44, 45, 46].

The existence of almost periodic, almost automorphic, pseudo-almost periodic, and pseudo-almost automorphic constitutes one of the most attractive topics in qualitative theory of differential equations due to their applications. Some contributions on (weighted) pseudo-almost periodic solutions to abstract differential and partial differential equations have recently been made, among them are [6, 10, 13, 15, 16, 18, 19, 20, 30, 44, 45, 46]. However, the existence of weighted pseudo-almost periodic solutions to the partial evolution equations of the form Eq. (1.2) is an untreated original topic, which in fact is the main motivation of the present paper.

The paper is organized as follows: Section 2 is devoted to preliminaries facts related to the existence of an evolution family. Some preliminary results on intermediate spaces are also stated there. In addition, basic definitions and results on the concept of pseudo-almost periodic functions are given. In Section 3, we first state and prove a key technical lemma (Lemma 3.2); next we prove the main result.

2. Preliminaries

This section is devoted to some preliminary results needed in the sequel. We basically use the same setting as in [7] with slight adjustments.

In this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space, $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the so-called P. Acquistapace and B. Terreni conditions (Assumption (H.1)). Moreover, the operators $A(t)$ are not necessarily densely defined. The (possibly unbounded) linear operators $B(t), C(t)$ are defined on $\mathbb{X}$ such that the family of operators $A(t)+B(t)+C(t)$ are not trivial for each $t \in \mathbb{R}$.

The functions, $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta^t$ $(0 < \alpha < \beta < 1)$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are respectively jointly continuous satisfying some additional assumptions.

If L is a linear operator on $\mathbb{X}$, then:

- $\rho(L)$ stands for the resolvent of L,
- $\sigma(L)$ stands for the spectrum of L,
- $D(L)$ stands for the domain of L,
- $N(L)$ stands for the nullspace of L, and
- $R(L)$ stands for the range of L.

Moreover, one defines

$$R(\lambda, L) := (\lambda I - L)^{-1}$$

for all $\lambda \in \rho(A)$.

Throughout the rest of the paper, we set $Q(s) = I - P(s)$ for a family of projections $P(s)$ with $s \in \mathbb{R}$. The space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{Z}$ equipped with its natural topology. When $\mathbb{Y} = \mathbb{Z}$, then this is simply denoted by $B(\mathbb{Y})$.

2.1. **Evolution Families.** (H.1). The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called

Acquistapace-Terreni conditions, that is, there exist constants $\omega \in \mathbb{R}$, $\theta \in \Big(\frac{\pi}{2}, \pi\Big)$, $K, L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$S_\theta \cup \Big\{0\Big\} \subset \rho\Big(A(t) - \omega\Big) \ni \lambda, \qquad \Big\|R\Big(\lambda, A(t) - \omega\Big)\Big\| \leq \frac{K}{1 + \Big|\lambda\Big|} \tag{2.1}$$

and

$$\Big\|\Big(A(t) - \omega\Big)R\Big(\lambda, A(t) - \omega\Big)\Big[R\Big(\omega, A(t)\Big) - R\Big(\omega, A(s)\Big)\Big]\Big\| \leq L\Big|t - s\Big|^\mu \Big|\lambda\Big|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \Big\{\lambda \in \mathbb{C} \setminus \{0\} : \Big|\arg \lambda\Big| \leq \theta\Big\}$.

Note that in the particular case when $A(t)$ has a constant domain $D = D(A(t))$, it is well-known [4, 36] that Eq. (2.2) can be replaced with the following: There exist constants L and $0 < \gamma \leq 1$ such that

$$\Big\|\Big(A(t) - A(s)\Big)R\Big(\omega, A(r)\Big)\Big\| \leq L\Big|t - s\Big|^\gamma, \; s, t, r \in \mathbb{R}. \tag{2.3}$$

It should mentioned that (H.1) was introduced in the literature by Acquistapace and Terreni in [2, 3] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family

$$\mathbb{U} = \{U(t,s) : t, s \in \mathbb{R} \;\text{ such that }\; t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t,s)\mathbb{X} \subset D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t, s, r \in \mathbb{R}$ such that $t \geq s \geq r$;
(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$;
(c) $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is continuous for $t > s$;
(d) $U(\cdot, s) \in C^1((s,\infty), B(\mathbb{X}))$, $\dfrac{\partial U}{\partial t}(t,s) = A(t)U(t,s)$ and

$$\Big\|A(t)^k U(t,s)\Big\| \leq K\,(t-s)^{-k}$$

for $0 < t - s \leq 1$, $k = 0, 1$; and
(e) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

It should also be mentioned that the above-mentioned proprieties were mainly established in [1, Theorem 2.3] and [42, Theorem 2.1], see also [3, 41]. In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

Definition 2.1. One says that an evolution family $\mathbb{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ $(t \in \mathbb{R})$ that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;
(g) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$); and
(h) $\Big\|U(t,s)P(s)\Big\| \leq Ne^{-\delta(t-s)}$ and $\Big\|\widetilde{U}_Q(s,t)Q(t)\Big\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

2.2. **Interpolation Spaces.** This setting requires some estimates related to $U(t,s)$. For that, we make extensive use of the real interpolation spaces of order (α,∞) between $\mathbb{X}$ and $D(A(t))$, where $\alpha \in (0,1)$. We refer the reader to the following excellent books [4], [23], and [33] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (assumption (H.1) holds when $A(t)$ is replaced with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \Big\{ x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha \Big(A-\omega\Big) R\Big(r, A-\omega\Big)x\| < \infty \Big\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \|x\|_0^A := \|x\|,\ \mathbb{X}_1^A := D(A)$$

and $\|x\|_1^A := \|(\omega - A)x\|$. Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we will frequently be using the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega-A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \subset \mathbb{X}, \tag{2.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.5}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.4) hold with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ ([33, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \le c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.6}$$

We have the following fundamental estimates for the evolution family $\mathbb{U}$. Its proof was given in [7] though for the sake of clarity, we reproduce it here.

Proposition 2.2. *For $x \in \mathbb{X}$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) *There is a constant $c(\alpha)$, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.7}$$

(ii) *There is a constant $m(\alpha)$, such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.8}$$

Proof. (i) Using (2.6) we obtain

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\le r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\le r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)U(t-1,s)P(s)x\|^\alpha \\
&\le r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)\|^\alpha\|U(t-1,s)P(s)x\|^\alpha \\
&\le c_0(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\|
\end{aligned}$$

for $t-s \geq 1$ and $x \in \mathbb{X}$.

Since $(t-s)^{\alpha} e^{-\frac{\delta}{2}(t-s)} \to 0$ as $t \to +\infty$ it easily follows that there exists $c_1(\alpha) > 0$ such that

$$\|U(t,s)P(s)x\|_{\alpha}^{t} \quad \leq \quad c_1(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|x\|.$$

If $0 < t-s \leq 1$, we have

$$\begin{aligned}
\|U(t,s)P(s)x\|_{\alpha}^{t} &\leq r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^{\alpha} \\
&\leq r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\|^{\alpha} \\
&\leq r(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})\|^{\alpha}\|U(\frac{t+s}{2},s)P(s)x\|^{\alpha} \\
&\leq c_2(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|,
\end{aligned}$$

and hence

$$\|U(t,s)P(s)x\|_{\alpha}^{t} \quad \leq \quad c(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|x\| \;\text{ for }\; t > s.$$

(ii)

$$\begin{aligned}
\|\widetilde{U}_Q(s,t)Q(t)x\|_{\alpha}^{s} &\leq r(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)\widetilde{U}_Q(s,t)Q(t)x\|^{\alpha} \\
&\leq r(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\widetilde{U}_Q(s,t)Q(t)x\|^{\alpha} \\
&\leq r(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^{\alpha}\|\widetilde{U}_Q(s,t)Q(t)x\|^{\alpha} \\
&\leq r(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^{\alpha}e^{-\delta(t-s)\alpha}\|x\| \\
&\leq m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}$$

In the last inequality we have used that $\|A(s)Q(s)\| \leq c$ for some constant $c \geq 0$, see e.g. [40, Proposition 3.18]. □

In addition to above, we also need the following assumptions:

(H.2). There exists $0 \leq \alpha < \beta < 1$ such that

$$\mathbb{X}_{\alpha}^{t} = \mathbb{X}_{\alpha} \;\text{ and }\; \mathbb{X}_{\beta}^{t} = \mathbb{X}_{\beta}$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

If $0 \leq \alpha < \beta < 1$, then we let $k(\alpha)$ denote the bound of the embedding $\mathbb{X}_{\beta} \hookrightarrow \mathbb{X}_{\alpha}$, that is,

$$\|u\|_{\alpha} \leq k(\alpha)\|u\|_{\beta}$$

for each $u \in \mathbb{X}_{\beta}$.

Hypothesis (H.3). The evolution family $\mathbb{U}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$. Moreover, $0 \in \rho(A(t))$ for each $t \in \mathbb{R}$ and the following holds

$$\sup_{t,s\in\mathbb{R}} \|A(s)A^{-1}(t)\|_{B(\mathbb{X},\mathbb{X}_{\alpha})} < c_0. \tag{2.9}$$

Remark 2.3. Note that Eq. (2.9) is satisfied in many cases in the literature. In particular, it holds when $A(t) = d(t)A$ where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is any closed linear operator such that $0 \in \rho(A)$ and $d : \mathbb{R} \mapsto \mathbb{R}$ with $\inf_{t\in\mathbb{R}} |d(t)| > 0$ and $\sup_{t\in\mathbb{R}} |d(t)| < \infty$.

2.3. **Weighted Pseudo Almost Periodic Functions.** Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ will be equipped with the sup. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

Let $\mathbb{U}$ denote the collection of all functions (weights) $\rho : \mathbb{R} \mapsto (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho(t) > 0$ for almost each $t \in \mathbb{R}$. Throughout the rest of the paper, if $\rho \in \mathbb{U}$ and $T > 0$, we use the notation

$$m(T, \rho) := \int_{-T}^{T} \rho(t)dt.$$

As in the particular case when $\rho(t) = 1$ for each $t \in \mathbb{R}$, we are exclusively interested in those weights, ρ, for which, $\lim_{T\to\infty} m(T, \rho) = \infty$. The notations $\mathbb{U}_\infty, \mathbb{U}_B$ stands for the sets of weights functions

$$\mathbb{U}_\infty := \left\{\rho \in \mathbb{U} : \lim_{T\to\infty} m(T, \rho) = \infty \text{ and } \liminf_{t\to\infty} \rho(t) > 0\right\},$$

$$\mathbb{U}_B := \{\rho \in \mathbb{U}_\infty : \ \rho \ \text{ is bounded}\}.$$

Obviously, $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.

Definition 2.4. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t + \tau) - f(t)\| < \varepsilon \ \text{ for each } \ t \in \mathbb{R}.$$

The number τ above is called an *ε-translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{X})$.

Definition 2.5. A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for each $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\|F(t + \tau, y) - F(t, y)\| < \varepsilon \ \text{ for each } \ t \in \mathbb{R}, \quad y \in K.$$

The collection of those functions is denoted by $AP(\mathbb{Y}, \mathbb{X})$.

To introduce those weighted pseudo almost periodic functions, we need to define the "weighted ergodic" space $PAP_0(\mathbb{X}, \rho)$. Weighted pseudo almost periodic functions will then appear as perturbations of almost periodic functions by elements of $PAP_0(\mathbb{X}, \rho)$.

Let $\rho \in \mathbb{U}_\infty$. Define

$$PAP_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{m(T, \rho)} \int_{-T}^{T} \|f(\sigma)\| \ \rho(\sigma)\, d\sigma = 0 \right\}.$$

Obviously, when $\rho(t) = 1$ for each $t \in \mathbb{R}$, one retrieves the so-called ergodic space of Zhang, that is, $PAP_0(\mathbb{X})$, defined by

$$PAP_0(\mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|f(\sigma)\|\, d\sigma = 0 \right\}.$$

Clearly, the spaces $PAP_0(\mathbb{X},\rho)$ are richer than $PAP_0(\mathbb{X})$ and give rise to an enlarged space of pseudo almost periodic functions. In Corollary 2.17, some sufficient conditions on the weight $\rho \in \mathbb{U}_\infty$ are given so that $PAP_0(\mathbb{X},\rho) = PAP_0(\mathbb{X})$.

In the same way, we define $PAP_0(\mathbb{Y},\mathbb{X},\rho)$ as the collection of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that $F(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{T\to\infty} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|F(s,y)\|\, \rho(s)\, ds = 0$$

uniformly in compact subset of $\mathbb{Y}$.

We are now ready to define the notion of weighted pseudo almost periodicity.

Definition 2.6. (Diagana) Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R},\mathbb{X})$ is called weighted pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X},\rho)$. The collection of such functions will be denoted by $PAP(\mathbb{X},\rho)$.

Remark 2.7.

(i) The functions g and ϕ appearing in Definition 2.6 are respectively called the almost periodic and the weighted ergodic perturbation components of f.

(ii) Let $\rho \in \mathbb{U}_\infty$ and assume that

$$\limsup_{s\to\infty} \left[\frac{\rho(s+\tau)}{\rho(s)}\right] < \infty \tag{2.10}$$

and

$$\limsup_{T\to\infty} \left[\frac{m(T+\tau,\rho)}{m(T,\rho)}\right] < \infty \tag{2.11}$$

for every $\tau \in \mathbb{R}$. In that case, the space $PAP(\mathbb{X},\rho)$ is translation invariant.

In this paper, all weights $\rho \in \mathbb{U}_\infty$ for which $PAP(\mathbb{X},\rho)$ is translation-invariant will be denoted $\mathbb{U}_0^{inv}$. Obviously, if ρ satisfies both Eq. (2.10) and Eq. (2.11), then $\rho \in \mathbb{U}_0^{inv}$.

Theorem 2.8. [31] *Fix $\rho \in \mathbb{U}_0^{inv}$. The decomposition of a weighted pseudo almost periodic function $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X},\rho)$, is unique.*

Lemma 2.9. *Let $\rho \in \mathbb{U}_0^{inv}$. Then the space $(PAP(\mathbb{X},\rho), \|\cdot\|_\infty)$ is a Banach space.*

Definition 2.10. (Diagana) A function $F \in BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is called weighted pseudo almost periodic if $F = G+\Phi$, where $G \in AP(\mathbb{X},\mathbb{Y},\rho)$ and $\Phi \in PAP_0(\mathbb{X},\mathbb{Y},\rho)$. The class of such functions will be denoted by $PAP(\mathbb{X},\mathbb{Y},\rho)$.

Proposition 2.11. *Let $\rho \in \mathbb{U}_0^{inv}$. Let $f \in PAP_0(\mathbb{R},\rho)$ and $g \in L^1(\mathbb{R})$. Then $f * g$, the convolution of f and g on $\mathbb{R}$, belongs to $PAP_0(\mathbb{R},\rho)$.*

Proof. From $f \in PAP_0(\mathbb{R},\rho)$ and $g \in L^1(\mathbb{R})$ it is clear that $f * g \in BC(\mathbb{R})$. Moreover, for $T > 0$ we see that

$$\begin{aligned}\frac{1}{m(T,\rho)} \int_{-T}^{T} |(f*g)(t)|\rho(t)dt &\leq \int_{-\infty}^{+\infty} |g(s)| \left(\frac{1}{m(T,\rho)} \int_{-T}^{T} |f(t-s)|\rho(t)dt\right) ds \\ &= \int_{-\infty}^{+\infty} |g(s)|\phi_T(s)ds,\end{aligned}$$

where $\phi_T(s) = \dfrac{1}{m(T,\rho)} \displaystyle\int_{-T}^{T} |f(t-s)|\rho(t)dt$.

Since $PAP_0(\mathbb{R},\rho)$ is translation invariant, it follows that $\phi_T(s) \mapsto 0$ as $T \mapsto \infty$. Next, using the boundedness of ϕ_T ($|\phi_T(s)| \leq \|f\|_\infty$) and the fact that $g \in L^1(\mathbb{R})$, the Lebesgue dominated convergence theorem yields

$$\lim_{T\to\infty} \int_{-\infty}^{+\infty} |g(s)|\phi_T(s)ds = 0,$$

which prove that $f * g \in PAP_0(\mathbb{R},\rho)$. $\square$

It is clear that if $h \in AP(\mathbb{R})$ and $\psi \in L^1(\mathbb{R})$, then the convolution $h*\psi \in AP(\mathbb{R})$. Combining those results one obtains.

Corollary 2.12. *Let $\rho \in \mathbb{U}_0^{inv}$ and let $f \in PAP(\mathbb{R},\rho)$ and $g \in L^1(\mathbb{R})$. Then $f * g$ belongs to $PAP(\mathbb{R},\rho)$.*

Example 2.13. Let $\rho \in \mathbb{U}_0^{inv}$. Define the function $W(\cdot)$ by

$$W(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy,$$

where $K \in L^1(\mathbb{R})$ and $f \in PAP(\mathbb{R},\rho)$. Then $W \in PAP(\mathbb{R},\rho)$, by Corollary 2.12.

If $f, g \in PAP(\mathbb{X},\rho)$ and let $\lambda \in \mathbb{R}$, then $f + \lambda g$ is also in $PAP(\mathbb{X},\rho)$. Moreover, if $|f(\cdot)|$ is not even and $t \to \frac{\rho(-t)}{\rho(t)} \in L^\infty(\mathbb{R})$ then the function $\tilde{f}(t) := f(-t)$ for $t \in \mathbb{R}$ is also in $PAP(\mathbb{X},\rho)$. In particular, if ρ is even, then $\tilde{f}$ belongs to $PAP(\mathbb{X},\rho)$.

Definition 2.14. Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. One says that ρ_1 is equivalent to ρ_2 and denote it $\rho_1 \prec \rho_2$, if the following limits

$$\liminf_{t\to\infty} \frac{\rho_1}{\rho_2}(t) \quad \text{and} \quad \limsup_{t\to\infty} \frac{\rho_2}{\rho_1}(t)$$

are finite.

Let $\rho_1, \rho_2, \rho_3 \in \mathbb{U}_\infty$. It is clear that $\rho_1 \prec \rho_1$ (reflexivity); if $\rho_1 \prec \rho_2$, then $\rho_2 \prec \rho_1$ (symmetry); and if $\rho_1 \prec \rho_2$ and $\rho_2 \prec \rho_3$, then $\rho_1 \prec \rho_3$ (transitivity). So, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$. The equivalence class of a given weight $\rho \in \mathbb{U}_\infty$ will be denoted by $\breve{\rho} = \{\varpi \in \mathbb{U}_\infty : \rho \prec \varpi\}$. It is then clear that

$$\mathbb{U}_\infty = \bigcup_{\rho\in\mathbb{U}_\infty} \breve{\rho}.$$

Theorem 2.15. *If $\rho_1, \rho_2 \in \breve{\rho}$, then $PAP_0(\mathbb{X},\rho_1) = PAP_0(\mathbb{X},\rho_2)$.*

Proof. From $\rho_1 \prec \rho_2$, there exist constants $K, K', T_0 > 0$ such that $K'\rho_2(t) \leq \rho_1(t) \leq K\rho_2(t)$ for each $|t| > T_0$. Consequently, for $T > T_0$,

$$\begin{aligned} m(T,\rho_1) = \int_{-T}^{T} \rho_1(s)ds &\leq \int_{-T}^{-T_0} \frac{\rho_1(s)}{\rho_2(s)}\rho_2(s)ds + \int_{-T_0}^{T_0} \rho_1(s)ds + \int_{T_0}^{T} \frac{\rho_1(s)}{\rho_2(s)}\rho_2(s)ds \\ &\leq K\int_{-T}^{-T_0} \rho_2(s)ds + \int_{-T_0}^{T_0} \rho_1(s)ds + K\int_{T_0}^{T} \rho_2(s)ds \\ &\leq \int_{-T_0}^{T_0} \rho_1(s)ds + Km(T,\rho_2), \end{aligned}$$

and hence

$$\frac{1}{m(T,\rho_2)} \leq \frac{K}{m(T,\rho_1) - \displaystyle\int_{-T_0}^{T_0} \rho_1(s)ds}, \qquad T \geq T_0.$$

Similarly, one can show that

$$\frac{1}{m(T,\rho_1)} \leq \frac{1}{K'(m(T,\rho_2) - \displaystyle\int_{-T_0}^{T_0} \rho_2(s)ds)}, \qquad T \geq T_0.$$

Let $\phi \in PAP_0(\mathbb{X},\rho_2)$. In view of the above it easily follows that for $T > T_0$

$$\begin{aligned}
&\frac{1}{m(T,\rho_1)} \int_{-T}^{T} \|\phi(s)\| \rho_1(s)ds \\
&= \frac{1}{m(T,\rho_1)} \int_{-T}^{-T_0} \|\phi(s)\| \left(\frac{\rho_1}{\rho_2}\right)(s)\rho_2(s)ds + \frac{1}{m(T,\rho_1)} \int_{-T_0}^{T_0} \|\phi(s)\| \rho_1(s)ds \\
&\quad + \frac{1}{m(T,\rho_1)} \int_{T_0}^{T} \|\phi(s)\| \left(\frac{\rho_1}{\rho_2}\right)(s)\rho_2(s)ds \\
&\leq \frac{K}{m(T,\rho_1)} \int_{-T}^{-T_0} \|\phi(s)\| \rho_2(s)ds + \frac{1}{m(T,\rho_1)} \int_{-T_0}^{T_0} \|\phi(s)\| \rho_1(s)ds \\
&\quad + \frac{K}{m(T,\rho_1)} \int_{T_0}^{T} \|\phi(s)\| \rho_2(s)ds \\
&\leq \frac{K}{K'\Big(m(T,\rho_2) - \displaystyle\int_{-T_0}^{T_0} \rho_2(s)ds\Big)} \int_{-T}^{T} \|\phi(s)\| \rho_2(s)ds + \frac{1}{m(T,\rho_1)} \int_{-T_0}^{T_0} \|\phi(s)\| \rho_1(s)ds,
\end{aligned}$$

which yields

$$\lim_{T\mapsto\infty} \frac{1}{m(T,\rho_1)} \int_{-T}^{T} \|\phi(s)\| \rho_1(s)ds = 0,$$

and hence $PAP_0(\mathbb{X},\rho_2) \subset PAP_0(\mathbb{X},\rho_1)$. Similarly, one can show that $PAP_0(\mathbb{X},\rho_1) \subset PAP_0(\mathbb{X},\rho_2)$. □

In view of the above, the proof of the next corollary is quite immediate.

Corollary 2.16. *If $\rho_1 \prec \rho_2$, then (i) $PAP(\mathbb{X},\rho_1+\rho_2) = PAP(\mathbb{X},\rho_1) = PAP(\mathbb{X},\rho_2)$, and (ii) $PAP(\mathbb{X},\frac{\rho_1}{\rho_2}) = PAP(\mathbb{X},\frac{\rho_2}{\rho_1}) = PAP(\mathbb{X},\breve{1}) = PAP(\mathbb{X})$.*

Another immediate consequence of Theorem 2.15 is that $PAP(\mathbb{X},\rho) = PAP(\mathbb{X},\breve{\rho})$. This enables us to identify the Zhang's space $PAP(\mathbb{X})$ with a weighted pseudo almost periodic class $PAP(\mathbb{X},\rho)$.

Corollary 2.17. *If $\rho \in \mathbb{U}_B$, then $PAP(\mathbb{X},\rho) = PAP(\mathbb{X},\breve{1}) = PAP(\mathbb{X})$.*

Theorem 2.18. *Let $\rho \in \mathbb{U}_\infty$, $F \in PAP(\mathbb{X},\mathbb{Y},\rho)$ and $h \in PAP(\mathbb{Y},\rho)$. Assume that there exists a function $L_F : \mathbb{R} \mapsto [0,\infty)$ satisfying*

$$(2.12) \quad \|F(t,z_1) - F(t,z_2)\|_{\mathbb{Y}} \;\leq\; L_F(t)\,\|z_1 - z_2\|\,, \quad \forall t \in \mathbb{R},\ \forall z_1, z_2 \in \mathbb{X}.$$

If $\sup_{t\in\mathbb{R}} L_F(t) = L_\infty < \infty$, then the function $t \mapsto F(t,h(t))$ belongs to $PAP(\mathbb{Y},\rho)$.

Proof. Assume that $F = F_1 + \varphi$, $h = h_1 + h_2$, where $F_1 \in AP(\mathbb{X}, \mathbb{Y})$, $\varphi \in PAP_0(\mathbb{X}, \mathbb{Y})$, $h_1 \in AP(\mathbb{X})$ and $h_2 \in PAP_0(\mathbb{X})$. Consider the decomposition

$$F(t, h(t)) \quad = \quad F_1(t, h_1(t)) + [F(t, h(t)) - F(t, h_1(t)))] + \varphi(t, h_1(t)).$$

Since $F_1(\cdot, h_1(\cdot)) \in AP(\mathbb{Y})$, it remains to prove that both $[F(\cdot, h(\cdot)) - F(\cdot, h_1(\cdot)))]$ and $\varphi(\cdot, h_1(\cdot))$ belong to $PAP_0(\mathbb{Y})$. Indeed, using the assumption on L_F it follows that

$$\begin{aligned}
\frac{1}{m(T,\rho)} \int_{-T}^{T} \|F(t, h(t)) - F(t, h_1(t)))\|_{\mathbb{Y}} \rho(t) dt &\leq \frac{1}{m(T,\rho)} \int_{-T}^{T} L_F(t) \|h_2(t)\| \rho(t) dt \\
&\leq \frac{L_\infty}{m(T,\rho)} \int_{-T}^{T} \|h_2(t)\| \rho(t) dt,
\end{aligned}$$

which implies that $[F(\cdot, h(\cdot)) - F(\cdot, h_1(\cdot)))] \in PAP_0(\mathbb{Y}, \rho)$.

Since $h_1(\mathbb{R})$ is relatively compact in $\mathbb{X}$ and F_1 is uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{X}$ is a compact subset, for $\varepsilon > 0$ there exists $0 < \delta \leq \varepsilon$ such that

$$\|F_1(t, z) - F_1(t, \bar{z})\|_{\mathbb{Y}} \leq \varepsilon, \qquad z, \bar{z} \in h_1(\mathbb{R})$$

for every $z, \bar{z} \in h_1(\mathbb{R})$ with $\|z - \bar{z}\| < \delta$. Now, fix $z_1, \ldots, z_n \in h_1(\mathbb{R})$ such that

$$h_1(\mathbb{R}) \subset \bigcup_{i=1}^{n} B_\delta(z_i, \mathbb{Z}).$$

Obviously, the sets $E_i = h_1^{-1}(B_\delta(z_i))$ form an open covering of $\mathbb{R}$, and therefore using the sets

$$B_1 = E_1, \quad B_2 = E_2 \setminus E_1, \quad \text{and} \quad B_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j,$$

one obtains a covering of $\mathbb{R}$ by disjoint open sets.

For $t \in B_i$ with $h_1(t) \in B_\delta(z_i)$

$$\begin{aligned}
\|\varphi(t, h_1(t))\|_{\mathbb{Y}} &\leq \|F(t, h_1(t)) - F(t, z_i)\|_{\mathbb{Y}} + \| - F_1(t, h_1(t)) + F_1(t, z_i)\|_{\mathbb{Y}} \\
&+ \| - \varphi(t, z_i)\|_{\mathbb{Y}} \\
&\leq L_F(t) \|h_1(t) - z_i\| + \varepsilon + \|\varphi(t, z_i)\|_{\mathbb{Y}} \\
&\leq L_F(t) \varepsilon + \varepsilon + \|\varphi(t, z_i)\|_{\mathbb{Y}}.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{m(T,\rho)} \int_{-T}^{T} \|\varphi(t, h_1(t))\|_{\mathbb{Y}} \rho(t) dt &\leq \frac{1}{m(T,\rho)} \sum_{i=1}^{n} \int_{B_i \cap [-T,T]} \|\varphi(t, h_1(t))\|_{\mathbb{Y}} \rho(t) dt \\
&\leq \frac{1}{m(T,\rho)} \int_{-T}^{T} \left[\sup_{t \in \mathbb{R}} L_F(t) \varepsilon + \varepsilon\right] \rho(t) dt \\
&+ \sum_{i=1}^{n} \frac{1}{m(T,\rho)} \int_{-T}^{T} \|\varphi(t, z_j)\|_{\mathbb{Y}} \rho(t) dt.
\end{aligned}$$

In view of the above it is clear that $\varphi(\cdot, h_1(\cdot))$ belongs to $PAP_0(\mathbb{Y}, \rho)$. $\square$

Corollary 2.19. *Let $f \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ satisfying the Lipschitz condition*

$$\|f(t, u) - f(t, v)\| \leq L \|u - v\|_{\mathbb{Y}} \quad \textit{for all} \quad u, v \in \mathbb{Y}, \ t \in \mathbb{R}.$$

(a) *If* $h \in PAP(\mathbb{Y}, \breve{\rho})$*, then* $f(\cdot, h(\cdot)) \in PAP(\mathbb{X}, \breve{\rho})$*.*
(b) *Let* $\rho_1, \rho_2 \in \mathbb{U}_\infty$ *with* $\rho_2 \in \breve{\rho}_1$*. If* $f \in PAP(\mathbb{Y}, \mathbb{X}, \breve{\rho}_1)$ *and* $h \in PAP(\mathbb{Y}, \rho_2)$*, then* $f(\cdot, h(\cdot)) \in PAP_0(\mathbb{X}, \breve{\rho}_1)$*.*
(c) *If* $\rho \in \mathbb{U}_B$*,* $f \in PAP(\mathbb{Y}, \mathbb{X}, \breve{\rho})$ *and* $h \in PAP(\mathbb{Y}, \rho)$*, then* $f(\cdot, h(\cdot)) \in PAP(\mathbb{X})$*.*

3. Main results

Throughout the rest of the paper we denote by $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, the nonlinear integral operators defined by

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s, B(s)u(s))ds,$$

$$(\Gamma_2 u)(t) := \int_{t}^{\infty} A(s)\widetilde{U}_Q(t,s)Q(s)f(s, B(s)u(s))ds,$$

$$(\Gamma_3 u)(t) := \int_{-\infty}^{t} U(t,s)P(s)g(s, C(s)u(s))ds, \text{ and}$$

$$(\Gamma_4 u)(t) := \int_{t}^{\infty} \widetilde{U}_Q(t,s)Q(s)g(s, C(s)u(s))ds.$$

Moreover, we suppose that the linear operators $B(t), C(t) : \mathbb{X}_\alpha \mapsto \mathbb{X}$ for all $t \in \mathbb{R}$, are bounded and set

$$\varpi := \max\left(\sup_{t\in\mathbb{R}} \|B(t)\|_{B(\mathbb{X}_\alpha, \mathbb{X})}, \ \sup_{t\in\mathbb{R}} \|C(t)\|_{B(\mathbb{X}_\alpha, \mathbb{X})} \right).$$

Furthermore, $t \mapsto B(t)u$ and $t \mapsto C(t)u$ are almost periodic for each $u \in \mathbb{X}_\alpha$.

To study Eq. (1.2), in addition to the previous assumptions, we require the following additional assumptions:

(H.4) $R(\omega, A(\cdot)) \in AP(B(\mathbb{X}_\alpha))$. Moreover, there exists a function $H : [0, \infty) \mapsto [0, \infty)$ with $H \in L^1[0, \infty)$ such that for every $\varepsilon > 0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a τ with the property

$$\|A(t+\tau)U(t+\tau, s+\tau) - A(t)U(t,s)\|_{B(\mathbb{X}, \mathbb{X}_\alpha)} \leq \varepsilon H(t-s)$$

for all $t, s \in \mathbb{R}$ with $t > s$.

(H.5) Let $\rho \in \mathbb{U}_0^{inv}$ and let $0 < \alpha < \beta < 1$. We suppose $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta$ belongs to $PAP(\mathbb{X}, \mathbb{X}_\beta, \rho)$ while $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ belongs to $PAP(\mathbb{X}, \mathbb{X}, \rho)$. Moreover, the functions f, g are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K > 0$ such that

$$\|f(t,u) - f(t,v)\|_\beta \leq K\|u - v\|,$$

and

$$\|g(t,u) - g(t,v)\| \leq K\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

To study the existence and uniqueness of pseudo almost periodic solutions to Eq. (1.2) we first introduce the notion of mild solution, which has been adapted from Diagana et al [13, Defintion 3.1].

Definition 3.1. A function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to Eq. (1.2) provided that the function $s \to A(s)U(t,s)P(s)f(s, B(s)u(s))$ is integrable on (s,t), $s \to A(s)U(t,s)Q(s)f(s, B(s)u(s))$ is integrable on (t,s) and

$$\begin{aligned}
u(t) &= -f(t, B(t)u(t)) + U(t,s)\Big(u(s) + f(s, B(s)u(s))\Big) \\
&- \int_s^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds + \int_t^s A(s)\widetilde{U}_Q(t,s)Q(s)f(s, B(s)u(s))ds \\
&+ \int_s^t U(t,s)P(s)g(s, C(s)u(s))ds - \int_t^s \widetilde{U}_Q(t,s)Q(s)g(s, C(s)u(s))ds
\end{aligned}$$

for $t \geq s$ and for all $t, s \in \mathbb{R}$.

Under assumptions (H.1)-(H.2)-(H.3)-(H.5), it can be easily shown any mild solution to Eq. (1.2) satisfies

$$\begin{aligned}
u(t) &= -f(t, B(t)u(t)) - \int_{-\infty}^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds \\
&+ \int_t^\infty A(s)\widetilde{U}_Q(t,s)Q(s)f(s, B(s)u(s))ds + \int_{-\infty}^t U(t,s)P(s)g(s, C(s)u(s))ds \\
&- \int_t^\infty \widetilde{U}_Q(t,s)Q(s)g(s, C(s)u(s))ds
\end{aligned}$$

for each $\forall t \in \mathbb{R}$.

The proof of our main result requires the following key technical Lemma.

Lemma 3.2. *Under assumptions* (H.1)—(H.3)*, if* $0 \leq \mu < \alpha < \beta < 1$ *with* $2\alpha > \mu + 1$*, then there exist two constants* $m(\alpha, \beta), n(\alpha, \mu) > 0$ *such that*

$$\|A(s)\widetilde{U}_Q(t,s)Q(s)x\|_\alpha \leq m(\alpha,\beta)e^{\delta(s-t)}\|x\|_\beta \qquad \text{for } t \leq s, \tag{3.1}$$

$$\|A(s)U(t,s)P(s)x\|_\alpha \leq n(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta, \qquad \text{for } t > s. \tag{3.2}$$

Proof. Let $x \in \mathbb{X}_\beta$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\|A(s)\widetilde{U}_Q(t,s)Q(s)x\|_\alpha &\leq ck(\alpha)\,\|\widetilde{U}_Q(t,s)Q(s)x\|_\beta \\
&\leq ck(\alpha)m(\beta)e^{\delta(s-t)}\,\|x\| \\
&\leq m(\alpha,\beta)e^{\delta(s-t)}\|x\|_\beta
\end{aligned}$$

for $t \leq s$ by using Eq. (2.8).

Similarly, for each $x \in \mathbb{X}_\beta$, using Eq. (2.9), we obtain

$$\begin{aligned}
\|A(s)U(t,s)P(s)x\|_\alpha &= \|A(s)A(t)^{-1}A(t)U(t,s)P(s)x\|_\alpha \\
&\leq \|A(s)A(t)^{-1}\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|A(t)U(t,s)P(s)x\| \\
&\leq c_0\|A(t)U(t,s)P(s)x\| \\
&\leq c_0k\|A(t)U(t,s)P(s)x\|_\alpha
\end{aligned}$$

for $t \geq s$.

First of all, note that $\|A(t)U(t,s)\|_{B(\mathbb{X},\mathbb{X}_\alpha)} \leq N'(t-s)^{-(1-\alpha)}$ for all t, s such that $0 < t - s \leq 1$ and $\alpha \in [0,1]$.

Letting $t - s \geq 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)P(s)x\|_\alpha &= \|A(t)U(t,t-1)U(t-1,s)P(s)x\|_\alpha \\
&\leq \|A(t)U(t,t-1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|U(t-1,s)P(s)x\| \\
&\leq NN'e^{\delta}e^{-\delta(t-s)}\|x\| \\
&\leq K_1 e^{-\delta(t-s)}\|x\|_\beta \\
&= K_1 e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta.
\end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha} \to 0$ as $t \to \infty$ it follows that there exists $c_4(\alpha) > 0$ such that

$$\|A(t)U(t,s)P(s)x\|_\alpha \leq c_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta$$

and hence

$$\|A(s)U(t,s)P(s)x\|_\alpha \leq c_0 k c_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta$$

for all $t, s \in \mathbb{R}$ such that $t - s > 1$.

Now, let $0 < t - s \leq 1$. Using Eq. (2.7) and the fact $2\alpha > \mu + 1$, we obtain

$$\begin{aligned}
\|A(t)U(t,s)P(s)x\|_\alpha &= \|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\|_\alpha \\
&\leq \|A(t)U(t,\frac{t+s}{2})\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|U(\frac{t+s}{2},s)P(s)x\| \\
&\leq k_2\|A(t)U(t,\frac{t+s}{2})\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|U(\frac{t+s}{2},s)P(s)x\|_\mu \\
&\leq k_2 N'\Big(\frac{t-s}{2}\Big)^{\alpha-1}c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&\leq k_2 k' N'\Big(\frac{t-s}{2}\Big)^{\alpha-1}c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta \\
&\leq c_5(\alpha,\mu)(t-s)^{\alpha-1-\mu}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta \\
&\leq c_5(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta.
\end{aligned}$$

Therefore there exists $n(\alpha, \mu) > 0$ such that

$$\|A(t)U(t,s)P(s)x\|_\alpha \leq n(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\|_\beta$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. $\square$

In the rest of the paper we suppose that α, β, μ are given real numbers such that $0 \leq \mu < \alpha < \beta < 1$ with $2\alpha > \mu + 1$.

Lemma 3.3. *Under previous assumptions, if $\rho \in \mathbb{U}_0^{inv}$ and $u \in PAP(\mathbb{X}_\alpha, \rho)$, then $C(\cdot)u(\cdot) \in PAP(\mathbb{X}, \rho)$. Similarly, $B(\cdot)u(\cdot) \in PAP(\mathbb{X}, \rho)$.*

Proof. Let $u \in PAP(\mathbb{X}_\alpha, \rho)$ and suppose $u = u_1 + u_2$ where $u_1 \in AP(\mathbb{X}_\alpha)$ and $u_2 \in PAP_0(\mathbb{X}_\alpha, \rho)$. Then, $C(t)u(t) = C(t)u_1(t) + C(t)u_2(t)$ for all $t \in \mathbb{R}$. Since $u_1 \in AP(\mathbb{X}_\alpha)$, for every $\varepsilon > 0$ there exists $T_0(\varepsilon)$ such that every interval of length $T_0(\varepsilon)$ contains a τ such

$$\Big\|u_1(t+\tau) - u_1(t)\Big\|_\alpha < \frac{\varepsilon}{2}, \quad t \in \mathbb{R}.$$

Similarly, since $C(t) \in AP(B(\mathbb{X}_\alpha, \mathbb{X}))$, we have

$$\Big\|C(t+\tau) - C(t)\Big\|_{B(\mathbb{X}_\alpha,\mathbb{X})} < \frac{\varepsilon}{2}, \quad t \in \mathbb{R}.$$

Now

$$\begin{aligned}
\|C(t+\tau)u_1(t+\tau)-C(t)u_1(t)\| &= \\
\|C(t+\tau)u_1(t+\tau)-C(t)u_1(t+\tau)+C(t)u_1(t+\tau)-C(t)u_1(t)\| &\leq \\
\|[C(t+\tau)-C(t)]u_1(t+\tau)\|+\|C(t)[u_1(t+\tau)-u_1(t)]\| &\leq \\
\|[C(t+\tau)-C(t)]\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|u_1(t+\tau)\|_\alpha+\|C(t)\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|[u_1(t+\tau)-u_1(t)]\|_\alpha &\leq \\
\Big(\sup_{t\in\mathbb{R}}\|u_1(t)\|_\alpha+\varpi\Big)\frac{\varepsilon}{2},
\end{aligned}$$

and hence $t\mapsto C(t)u_1(t)$ belongs to $AP(\mathbb{X})$.

To complete the proof, it suffices to notice that

$$\frac{1}{m(T,\rho)}\int_{-T}^{T}\Big\|C(t)u_2(t)\Big\|dt\leq\frac{\varpi}{m(T,\rho)}\int_{-T}^{T}\|u_2(t)\|_\alpha dt$$

and hence

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\Big\|C(t)u_2(t)\Big\|dt=0.$$

□

Lemma 3.4. *Under assumptions* (H.1)—(H.5)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $PAP(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u\in PAP(\mathbb{X}_\alpha,\rho)$. From Lemma 3.3 it follows that $C(\cdot)u(\cdot)\in PAP(\mathbb{X},\rho)$. Setting $h(t)=g(t,Cu(t))$ and using Theorem 2.18 it follows that $h\in PAP(\mathbb{X},\rho)$. Now write $h=\phi+\zeta$ where $\phi\in AP(\mathbb{X})$ and $\zeta\in PAP_0(\mathbb{X},\rho)$.

Now $\Gamma_3 u$ can be rewritten as

$$(\Gamma_3u)(t) \;=\; \int_{-\infty}^{t}U(t,s)P(s)\phi(s)ds+\int_{-\infty}^{t}U(t,s)P(s)\zeta(s)ds.$$

Let

$$\Phi(t)=\int_{-\infty}^{t}U(t,s)P(s)\phi(s)ds,\quad\text{and}\quad\Psi(t)=\int_{-\infty}^{t}U(t,s)P(s)\zeta(s)ds$$

for each $t\in\mathbb{R}$.

The next step consists of showing that $\Phi\in AP(\mathbb{X}_\alpha)$ and $\Psi\in PAP_0(\mathbb{X}_\alpha,\rho)$. Obviously, $\Phi\in AP(\mathbb{X}_\alpha)$. Indeed, since $\phi\in AP(\mathbb{X})$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for every interval of length $l(\varepsilon)$ contains a τ with the property

$$\|\phi(t+\tau)-\phi(t)\|<\varepsilon\mu\quad\text{for each } t\in\mathbb{R},$$

where $\mu=\dfrac{\delta^{1-\alpha}}{c(\alpha)2^{1-\alpha}\Gamma(1-\alpha)}$.

Now

$$\begin{aligned}
\Phi(t+\tau)-\Phi(t) &= \int_{-\infty}^{t+\tau} U(t+\tau,s)P(s)\phi(s)ds - \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau,s+\tau)P(s+\tau)\phi(s+\tau)ds - \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau,s+\tau)P(s+\tau)\Big(\phi(s+\tau)-\phi(s)\Big)ds \\
&+ \int_{-\infty}^{t} \Big(U(t+\tau,s+\tau)P(s+\tau)-U(t,s)P(s)\Big)\phi(s)ds.
\end{aligned}$$

Using [8, 34] it follows that

$$\|\int_{-\infty}^{t} \Big(U(t+\tau,s+\tau)P(s+\tau)-U(t,s)P(s)\Big)\phi(s)ds\|_\alpha \le \frac{2\|\phi\|_\infty}{\delta}\varepsilon.$$

Similarly, Eq. (2.7) yields

$$\|\int_{-\infty}^{t} U(t+\tau,s+\tau)P(s+\tau)\Big(\phi(s+\tau)-\phi(s)\Big)ds\|_\alpha \le \varepsilon.$$

Therefore,

$$\|\Phi(t+\tau)-\Phi(t)\|_\alpha \le \left(1+\frac{2\|\phi\|_\infty}{\delta}\right)\varepsilon \text{ for each } t\in\mathbb{R},$$

and hence, $\Phi \in AP(\mathbb{X}_\alpha)$.

To complete the proof for Γ_3, we have to show that $\Psi \in PAP_0(\mathbb{X}_\alpha,\rho)$. First, note that $s\mapsto \Psi(s)$ is a bounded continuous function. It remains to show that

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\Psi(t)\|_\alpha dt = 0.$$

Again using Eq. (2.7) it follows that

$$\begin{aligned}
\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\Psi(t)\|_\alpha\rho(t)dt &\le \frac{c(\alpha)}{m(T,\rho)}\int_{-T}^{T}\int_{0}^{+\infty} s^{-\alpha}e^{-\frac{\delta}{2}s}\|\zeta(t-s)\|\rho(t)dsdt \\
&\le c(\alpha)\int_{0}^{+\infty} s^{-\alpha}e^{-\frac{\delta}{2}s}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\zeta(t-s)\|\rho(t)dtds.
\end{aligned}$$

Set

$$\Gamma_s(T) = \frac{1}{m(T,\rho)}\int_{-T}^{T}\|\zeta(t-s)\|\rho(t)dt.$$

Since $PAP_0(\mathbb{X},\rho)$ ($\rho\in\mathbb{U}_0^{inv}$) is translation invariant, then $t\mapsto\zeta(t-s)$ belongs to $PAP_0(\mathbb{X},m(T,\rho))$ for each $s\in\mathbb{R}$, and hence

$$\lim_{T\mapsto\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|\zeta(t-s)\|\rho(t)dt = 0$$

for each $s\in\mathbb{R}$.

One completes the proof by using the well-known Lebesgue dominated convergence theorem and the fact $\Gamma_s(T)\mapsto 0$ as $T\to\infty$ for each $s\in\mathbb{R}$.

The proof for $\Gamma_4 u(\cdot)$ is similar to that of $\Gamma_3 u(\cdot)$. However one makes use of Eq. (2.8) rather than Eq. (2.7). $\square$

Lemma 3.5. *Under assumptions* (H.1)—(H.5)*, the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAP(\mathbb{X}_\alpha, \rho)$ *into itself.*

Proof. Let $u \in PAP(\mathbb{X}_\alpha, \rho)$. From Lemma 3.3 it follows that the function $t \mapsto B(t)u(t)$ belongs to $PAP(\mathbb{X})$. Again, using Theorem 2.18 it follows that $\psi(\cdot) = f(\cdot, Bu(\cdot))$ is in $PAP(\mathbb{X}_\beta, \rho)$ whenever $u \in PAP(\mathbb{X}_\alpha, \rho)$. In particular,

$$\|\psi\|_{\infty,\beta} = \sup_{t\in\mathbb{R}} \|f(t, Bu(t))\|_\beta < \infty.$$

Now write $\psi = \phi + z$, where $w \in AP(\mathbb{X}_\beta)$ and $z \in PAP_0(\mathbb{X}_\beta, \rho)$, that is, $\Gamma_1 \psi = \Xi(\phi) + \Xi(z)$ where

$$\Xi w(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds, \quad \text{and}$$

$$\Xi z(t) := \int_{-\infty}^{t} A(s)U(t,s)P(s)z(s)ds.$$

Clearly, $\Xi(\phi) \in AP(\mathbb{X}_\alpha)$. Indeed, since $\phi \in AP(\mathbb{X}_\beta)$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a τ with the property

$$\|\phi(t+\tau) - \phi(t)\|_\beta < \varepsilon\nu \ \text{ for each } t \in \mathbb{R}$$

where $\nu = \dfrac{\delta^{1-\alpha}}{n(\alpha,\mu)4^{1-\alpha}\Gamma(1-\alpha)}$.

$$\begin{aligned}
\Xi\phi(t+\tau) - \Xi\phi(t) &= \int_{-\infty}^{t+\tau} A(s)U(t+\tau,s)P(s)\phi(s)ds - \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds \\
&= \int_{-\infty}^{t} A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)\Big(\phi(s+\tau) - \phi(s)\Big)ds \\
&+ \int_{-\infty}^{t} \Big(A(s+\tau)U(t+\tau,s+\tau)P(s+\tau) - A(s)U(t,s)P(s)\Big)\phi(s)ds.
\end{aligned}$$

Using Eq. (3.2) it follows that

$$\|\int_{-\infty}^{t} A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)\Big(\phi(s+\tau) - \phi(s)\Big)ds\|_\alpha \le \varepsilon.$$

Similarly, using assumption (H.4), it follows that

$$\|\int_{-\infty}^{t} \Big(A(s+\tau)U(t+\tau,s+\tau)P(s+\tau) - A(s)U(t,s)P(s)\Big)\phi(s)ds\|_\alpha \le \varepsilon N'\|H\|_{L^1}\|\phi\|_\infty$$

where $\|H\|_{L^1} = \displaystyle\int_0^\infty H(s)ds < \infty$.

Therefore,

$$\|\Xi(\phi)(t+\tau) - \Xi(\phi)(t)\|_\alpha \le \Big(1 + N'\|H\|_{L^1}\|\phi\|_\infty\Big)\varepsilon$$

for each $t \in \mathbb{R}$, and hence $\Xi(\phi) \in AP(\mathbb{X}_\alpha)$.

Now, let $T > 0$. Again from Eq. (3.2), we have

$$\begin{aligned}
\frac{1}{m(T,\rho)}\int_{-T}^{T}\|(\Xi z)(t)\|_\alpha\rho(t)dt &\leq \frac{1}{m(T,\rho)}\int_{-T}^{T}\int_0^{+\infty}\|A(s)U(t,s)P(s)z(t-s)\|_\alpha\rho(t)dsdt\\
&\leq \frac{n(\alpha,\mu)}{m(T,\rho)}\int_{-T}^{T}\int_0^{+\infty}s^{-\alpha}e^{-\frac{\delta}{4}s}\|z(t-s)\|_\beta\rho(t)dsdt\\
&\leq n(\alpha,\mu)\,.\int_0^{+\infty}s^{-\alpha}e^{-\frac{\delta}{4}s}\left(\frac{1}{m(T,\rho)}\int_{-T}^{T}\|z(t-s)\|_\beta\rho(t)dt\right)ds.
\end{aligned}$$

Now

$$\lim_{T\to\infty}\frac{1}{m(T,\rho)}\int_{-T}^{T}\|z(t-s)\|_\beta\rho(t)dt=0,$$

as $t\mapsto z(t-s)\in PAP_0(\mathbb{X}_\beta,\rho)$ for every $s\in\mathbb{R}$ ($\rho\in\mathbb{U}_0^{inv}$). One completes the proof by using the Lebesgue's dominated convergence theorem.

The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$ except that one makes use of Eq. (3.1) instead of Eq. (3.2). □

Theorem 3.6. *Under assumptions* (H.1)—(H.5)*, the evolution equation* (1.2) *has a unique weighted pseudo almost periodic mild solution whenever K is small enough.*

Proof. Consider the nonlinear operator $\mathbb{M}$ defined on $PAP(\mathbb{X}_\alpha,\rho)$ by

$$\begin{aligned}
\mathbb{M}u(t) &= -f(t,B(t)u(t))-\int_{-\infty}^{t}A(s)U(t,s)P(s)f(s,B(s)u(s))ds\\
&+ \int_t^{\infty}A(s)\widetilde{U}_Q(t,s)Q(s)f(s,B(s)u(s))ds+\int_{-\infty}^{t}U(t,s)P(s)g(s,C(s)u(s))ds\\
&- \int_t^{\infty}\widetilde{U}_Q(t,s)Q(s)g(s,C(s)u(s))ds
\end{aligned}$$

for each $t\in\mathbb{R}$.

We have seen that for every $u\in PAP(\mathbb{X}_\alpha,\rho)$, $f(\cdot,B(\cdot)u(\cdot))\in PAP(\mathbb{X}_\beta,\rho)\subset PAP(\mathbb{X}_\alpha,\rho)$. In view of Lemma 3.4 and Lemma 3.5, it follows that $\mathbb{M}$ maps $PAP(\mathbb{X}_\alpha,\rho)$ into itself. To complete the proof one has to show that $\mathbb{M}$ has a unique fixed-point.

Let $v,w\in PAP(\mathbb{X}_\alpha,\rho)$

$$\begin{aligned}
\|\Gamma_1(v)(t)-\Gamma_1(w)(t)\|_\alpha &\leq n(\alpha)K\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|B(s)v(s)-B(s)w(s)\|ds\\
&\leq n(\alpha)K\varpi\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|v(s)-w(s)\|_\alpha ds\\
&\leq n(\alpha)K\varpi\|v-w\|_{\infty,\alpha}\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}ds\\
&= 2^{1-\alpha}\,\delta^{\alpha-1}\,n(\alpha)\,\Gamma(1-\alpha)K\,\varpi\,\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Now

$$\begin{aligned}
\|\Gamma_2(v)(t)-\Gamma_2(w)(t)\|_\alpha &\leq m(\alpha,\beta)\int_t^\infty \|f(s,B(s)v(s))-f(s,B(s)w(s))\|_\beta ds\\
&\leq m(\alpha,\beta)K\varpi\int_t^{+\infty} e^{\delta(t-s)}\|B(s)v(s)-B(s)w(s)\|ds\\
&\leq m(\alpha,\beta)K\varpi\int_t^{+\infty} e^{\delta(t-s)}\|v(s)-w(s)\|_\alpha ds\\
&\leq m(\alpha,\beta)K\varpi\|v-w\|_{\infty,\alpha}\int_t^{+\infty} e^{\delta(t-s)}ds\\
&= \delta^{-1}m(\alpha,\beta)K\varpi\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Now for Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned}
\|\Gamma_3(v)(t)-\Gamma_3(w)(t)\|_\alpha &\leq \int_{-\infty}^t \|U(t,s)P(s)\left[g(s,C(s)v(s))-g(s,C(s)w(s))\right]\|_\alpha ds\\
&\leq Kc(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|C(s)v(s)-C(s)w(s)\|ds\\
&\leq \varpi Kc(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|v(s)-w(s)\|_\alpha ds\\
&\leq K\varpi c(\alpha)2^{1-\alpha}\,\delta^{\alpha-1}\,\Gamma(1-\alpha)\|v-w\|_{\infty,\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\|\Gamma_4(v)(t)-\Gamma_4(w)(t)\|_\alpha &\leq \int_t^\infty m(\alpha)e^{\delta(t-s)}\|g(s,C(s)v(s))-g(s,C(s)w(s))\|ds\\
&\leq \int_t^\infty m(\alpha)Ke^{\delta(t-s)}\|C(s)v(s)-C(s)w(s)\|ds\\
&\leq \varpi m(\alpha)K\int_t^\infty e^{\delta(t-s)}\|v(s)-w(s)\|_\alpha ds\\
&\leq Km(\alpha)\varpi\|v-w\|_{\infty,\alpha}\int_t^{+\infty} e^{\delta(t-s)}ds\\
&= K\delta^{-1}\varpi m(\alpha)\|v-w\|_{\infty,\alpha}.
\end{aligned}$$

Combining previous approximations it follows that

$$\|\mathbb{M}v-\mathbb{M}w\|_{\infty,\alpha}\leq Kc(\alpha,\mu,\beta,\delta,\varpi)\,.\,\|v-w\|_{\infty,\alpha},$$

and hence if K is small enough, then Eq. (1.2) has a unique solution, which obviously is its only weighted pseudo almost periodic solution. □

Example 3.7. Let $\Omega\subset\mathbb{R}^N$ ($N\geq 1$) be an open bounded subset with regular boundary $\Gamma=\partial\Omega$ and let $\mathbb{X}=L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_{L^2(\Omega)}$.

Let $\rho_m(t)=(1+t^2)^m$ where $m\in\mathbb{N}$. It can be easily shown that $\rho_m\in\mathbb{U}_0^{inv}$ for each $m\in\mathbb{N}$ with $m=0$ corresponding to the classical pseudo almost periodicity. Here we illustrate our abstract result by studying the existence of ρ_m-pseudo-almost

periodic solutions to the nonautonomous heat equation with gradient coefficients

$$(3.3)\begin{cases} \dfrac{\partial}{\partial t}\Big[\varphi + F\Big(t, b(t,x)\nabla\varphi\Big)\Big] = a(t,x)\Delta\varphi + G\Big(t, c(t,x)\nabla\varphi\Big), & \text{in } \mathbb{R}\times\Omega \\ \\ \varphi = 0, & \text{on } \mathbb{R}\times\Gamma \end{cases}$$

where the coefficients $a, b, c : \mathbb{R}\times\Omega \mapsto \mathbb{R}$ are almost periodic, and $F, G : \mathbb{R}\times\mathbb{X}_\alpha \mapsto L^2(\Omega)$ are ρ_m-pseudo-almost periodic functions, where $\mathbb{X}_\alpha = (L^2(\Omega), \mathbb{H}_0^1(\Omega)\cap\mathbb{H}^2(\Omega))_{\alpha,\infty}$.

Define the linear operator appearing in Eq. (3.3) as follows:

$$A(t)u = a(t,x)\mathbf{\Delta}u \ \text{ for all } \ u \in D(A(t)) = \mathbb{H}_0^1(\Omega)\cap\mathbb{H}^2(\Omega),$$

where $a : \mathbb{R}\times\Omega \mapsto \mathbb{R}$, in addition of being almost periodic satisfies the following assumptions:

(H.6) $\inf_{t\in\mathbb{R}, x\in\Omega} a(t,x) = m_0 > 0$, and

(H.7) there exists $L > 0$ and $0 < \mu \leq 1$ such that

$$|a(t,x) - a(s,x)| \leq L|s-t|^\mu$$

for all $t, s \in \mathbb{R}$ uniformly in $x \in \Omega$.

First of all, note that in view of the above, $\sup_{t\in\mathbb{R}, x\in\Omega} a(t,x) < \infty$. Also, a classical example of a function a satisfying the above-mentioned assumptions is for instance

$$a_\gamma(t,x) = 3 + \sin|x|t + \sin\gamma|x|t,$$

where $|x| = (x_1^2 + ... + x_N^2)^{1/2}$ for each $x = (x_1, x_2, ..., x_N) \in \Omega$ and $\gamma \in \mathbb{R}\setminus\mathbb{Q}$.

Take $\alpha > 1/2$ and let β, μ such that $0 \leq \mu < \alpha < \beta$ with $2\alpha > \mu + 1$. Setting

$$B(t) = b(t,x)\nabla \ \text{ and } \ C(t) = c(t,x)\nabla$$

and using the embeddings Eq. (2.4), one can easily see that for each $u \in \mathbb{X}_\beta$,

$$\begin{aligned} \|B(t)u\|_{L^2(\Omega)} \leq \|b\|_\infty \|\nabla u\|_{L^2(\Omega)} &\leq \|b\|_\infty \|u\|_{H_0^1(\Omega)} \\ &= \|b\|_\infty \|u\|_{D((\omega - A(t))^\alpha)} \\ &\leq c\|b\|_\infty \|u\|_\beta \end{aligned}$$

and hence $B(t) \in B(\mathbb{X}_\beta, L^2(\Omega))$. Similarly, $C(t) \in B(\mathbb{X}_\beta, L^2(\Omega))$. Moreover, because of the almost periodicity of $t \mapsto b(t,x), c(t,x)$ uniformly in $x \in \Omega$, one can see that for each $u \in \mathbb{X}_\beta$,

$$t \mapsto B(t)u \ \text{ and } \ t \mapsto C(t)u$$

are almost periodic.

Under previous assumptions, it is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace-Terreni conditions. Moreover, it can be easily shown that

$$R\Big(\omega, a(\cdot,x)\mathbf{\Delta}\Big)\varphi = \frac{1}{a(\cdot,x)} R\Big(\frac{\omega}{a(\cdot,x)}, \mathbf{\Delta}\Big)\varphi \in AP(\mathbb{X}_\alpha)$$

for each $\varphi \in L^2(\Omega)$ with

$$\Big\| R\Big(\omega, a\mathbf{\Delta}\Big)\Big\|_{B(L^2(\Omega))} \leq \frac{const.}{|\omega|}.$$

Furthermore, assumptions (H.1)—(H.4) are fulfilled.

We require the following assumption:

(H.8) Let $0 < \mu < \alpha < \beta < 1$, and $F, G : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}_\beta$ be ρ_m-pseudo-almost periodic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}_\alpha$. Moreover, the functions F, G are globally Lipschitz with respect to the second argument in the following sense: there exists $K' > 0$ such that
$$\Big\|F(t,\varphi) - F(t,\psi)\Big\|_\beta \leq K'\Big\|\varphi - \psi\Big\|_{L^2(\Omega)},$$
and
$$\Big\|G(t,\varphi) - G(t,\psi)\Big\|_{L^2(\Omega)} \leq K'\Big\|\varphi - \psi\Big\|_{L^2(\Omega)}$$
for all $\varphi, \psi \in L^2(\Omega)$ and $t \in \mathbb{R}$.

We have

Theorem 3.8. *Under previous assumptions including* (H.6)-(H.8)*, then the reaction-diffusion equation Eq.* (3.3) *has a unique solution* $\varphi \in PAP(\rho_m, (L^2(\Omega), \mathbb{H}^1_0(\Omega) \cap \mathbb{H}^2(\Omega))_{\alpha,\infty})$ *whenever* K' *is small enough.*

Classical examples of the above-mentioned functions $F, G : \mathbb{R} \times \mathbb{X}_\alpha \mapsto L^2(\Omega)$ are given as follows:
$$F\Big(t, b(t,x)\varphi\Big) = \frac{Ke(t,x)}{1 + |\nabla\varphi|} \quad \text{and}$$
$$G\Big(t, c(t,x)\varphi\Big) = \frac{Kh(t,x)}{1 + |\nabla\varphi|}$$
where the functions $e, h : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ are ρ_m-pseudo-almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \Omega$.

In this particular case, the corresponding reaction-diffusion equation, that is,
$$\begin{cases} \dfrac{\partial}{\partial t}\left[\varphi + \dfrac{Ke(t,x)}{1 + |\nabla\varphi|}\right] = a(t,x)\Delta\varphi + \dfrac{Kh(t,x)}{1 + |\nabla\varphi|}, & \text{in } \mathbb{R} \times \Omega \\ \\ \varphi = 0, & \text{on } \ \mathbb{R} \times \Gamma \end{cases}$$
has a unique solution $\varphi \in PAP(\rho_m, (L^2(\Omega), \mathbb{H}^1_0(\Omega) \cap \mathbb{H}^2(\Omega))_{\alpha,\infty})$ whenever K is small enough.

References

1. P. Acquistapace, Evolution operators and strong solutions of abstract linear parabolic equations, Differential Integral Equations **1** (1988), 433–457.
2. P. Acquistapace, F. Flandoli, B. Terreni, Initial boundary value problems and optimal control for nonautonomous parabolic systems. SIAM J. Control Optim. **29** (1991), 89–118.
3. P. Acquistapace, B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.
4. H. Amann, *Linear and Quasilinear Parabolic Problems*, Birkhäuser, Berlin 1995.
5. B. Amir and L. Maniar, Existence and some asymptotic behaviors of solutions to semilinear Cauchy problems with non dense domain via extrapolation spaces, *Rend. Circ. Mat. Palermo* (2000) 481-496.
6. B. Amir and L. Maniar, Composition of Pseudo-Almost Periodic Functions and Cauchy Problems with Perator of Nondense Domain. *Ann. Math. Blaise Pascal* **6** (1999), no. 1, pp. 1–11.
7. M. Baroun, S. Boulite, T. Diagana, and L. Maniar, Almost periodic solutions to some semilinear non-autonomous thermoelastic plate equations. *J. Math. Anal. Appl.* **349**(2009), no. 1, 74–84.

8. M. Baroun, S. Boulite, G. M. N'Guérékata, L. Maniar, Almost automorphy of semilinear parabolic evolution equations. *Electron. J. Diff. Eqns.*, Vol. 2008(2008), No. 60, pp. 1-9.
9. C. Corduneanu, *Almost Periodic Functions*, 2nd Edition, Chelsea-New York, 1989.
10. C. Cuevas and M. Pinto, Existence and Uniqueness of Pseudo Almost Periodic Solutions of Semilinear Cauchy Problems with Nondense Domain, *Nonlinear Anal* **45**(2001), no. 1, Ser. A: Theory Methods, pp. 73–83.
11. G. Da Prato and P. Grisvard, Equations d'évolution abstraites non linéaires de type parabolique. *Ann. Mat. Pura Appl.* (4) **120** (1979), pp. 329–396.
12. W. Desch, R. Grimmer, Ronald, and W. Schappacher, Well-Posedness and Wave Propagation for a Class of Integro-differential Equations in Banach Space. *J. Differential Equations* **74** (1988), no. 2, pp. 391–411.
13. T. Diagana, E. Hernández, and M. Rabello, Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay. *Math. Comput. Modelling.* **45** (2007), Issues 9-10, pp. 1241-1252.
14. T. Diagana, Pseudo Almost Periodic Solutions to Some Differential Equations, *Nonlinear Anal* **60** (2005), no. 7, pp. 1277–1286.
15. T. Diagana, *Pseudo almost periodic functions in Banach spaces.* Nova Science Publishers, Inc., New York, 2007.
16. T. Diagana and E. Hernàndez M., Existence and Uniqueness of Pseudo Almost Periodic Solutions to Some Abstract Partial Neutral Functional-Differential Equations and Applications, *J. Math. Anal. Appl.* **327**(2007), no. 2, pp. 776–791.
17. T. Diagana, C. M. Mahop, and G. M. N'Guérékata, Pseudo Almost Periodic Solution to Some Semilinear Differential Equations, *Math. Comp. Modelling* **43** (2006), no. 1-2, pp. 89–96.
18. T. Diagana, C. M. Mahop, G. M. N'Guérékata, and B. Toni, Existence and Uniqueness of Pseudo Almost Periodic Solutions to Some Classes of Semilinear Differential Equations and Applications. *Nonlinear Anal.* **64** (2006), no. 11, pp. 2442–2453.
19. T. Diagana, Existence and Uniqueness of Pseudo Almost Periodic Solutions to Some Classes of Partial Evolution Equations. *Nonlinear Anal.* **66** (2007), no. 2, 384–395.
20. T. Diagana and G. M. N'Guérékata, Pseudo Almost Periodic Mild Solutions To Hyperbolic Evolution Equationa in Abstract Intermediate Banach Spaces. *Applicable Anal.* **85** (2006), Nos. 6-7, pp. 769–780.
21. T. Diagana, Existence of Weighted Pseudo Almost Periodic Solutions to Some Classes of Hyperbolic Evolution Equations. *J. Math. Anal. Appl.* **350** (2009), no. 1, 18-28.
22. T. Diagana, *An Introduction to Classical and p-adic Theory of Linear Operators and Applications*, Nova Science Publishers, New York, 2006.
23. K. J. Engel and R. Nagel, *One Parameter Semigroups for Linear Evolution Equations*, Graduate texts in Mathematics, Springer Verlag 1999.
24. A. M. Fink, *Almost Periodic Differential Equations,* Lecture Notes in Mathematics **377**, Springer-Verlag, New York-Berlin, 1974.
25. J. A. Goldstein, Convexity, Boundedness, and Almost Periodicity for Differential Equations in Hilbert Space. *Internat. J. Math. Math. Sci* **2** (1979), no. 1, pp. 1–13.
26. E. Hernández and H. R. Henríquez, Existence of Periodic Solutions of Partial neutral Functional Differential Equations with Unbounded Delay. *J. Math. Anal. Appl* **221** (1998), no. 2, pp. 499–522.
27. E. Hernández, Existence Results for Partial Neutral Integro-differential Equations with Unbounded Delay. *J. Math. Anal. Appl* **292** (2004), no. 1, pp. 194–210.
28. E. Hernández M., M. L. Pelicer, and J. P. C. dos Santos , Asymptotically Almost Periodic and Almost Periodic Solutions for a Class of Evolution Equations, *Electron. J. Diff. Eqns* **2004**(2004), no. 61, pp. 1–15.
29. Y. Hino, T. Naito, N. V. Minh, and J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces.* Stability and Control: Theory, Methods and Applications, **15**. Taylor and Francis, London, 2002.
30. H. X. Li, F. L. Huang, and J. Y. Li, Composition of Pseudo Almost-Periodic Functions and Semilinear Differential Equations. *J. Math. Anal. Appl* **255** (2001), no. 2, pp. 436–446.
31. J. Liang, T.-J. Xiao, and J. Zhang, Decomposition of weighted pseudo almost periodic functions. *Nonlinear Anal.* **73** (2010), no. 10, pp. 3456-3461.
32. J.-L. Lions and J. Peetre, Sur une classe d'espaces d'interpolation. *Inst. Hautes tudes Sci. Publ. Math.* no. **19** (1964), pp. 5–68.

33. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, PNLDE Vol. *16*, Birkhäauser Verlag, Basel, 1995.
34. L. Maniar, R. Schnaubelt, Almost periodicity of inhomogeneous parabolic evolution equations, Lecture Notes in Pure and Appl. Math. vol. 234, Dekker, New York (2003), 299-318.
35. M. G. Naso, A. Benabdallah, Thermoelastic plate with thermal interior control, Mathematical models and methods for smart materials (Cortona, 2001), 247–250, Ser. Adv. Math. Appl. Sci., 62, World Sci. Publ., River Edge, NJ, 2002.
36. A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.
37. J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser, 1993.
38. R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations, Forum Math. **11**(1999), 543-566.
39. R. Schnaubelt, Asymptotically autonomous parabolic evolution equations, J. Evol. Equ. **1** (2001), 19–37.
40. R. Schnaubelt, Asymptotic behavior of parabolic nonautonomous evolution equations, in: M. Iannelli, R. Nagel, S. Piazzera (Eds.), Functional Analytic Methods for Evolution Equations, in: Lecture Notes in Math., **1855**, Springer-Verlag, Berlin, 2004, 401–472.
41. A. Yagi, Parabolic equations in which the coefficients are generators of infinitely differentiable semigroups II, Funkcial. Ekvac. **33** (1990), 139–150.
42. A. Yagi, Abstract quasilinear evolution equations of parabolic type in Banach spaces, Boll. Un. Mat. Ital. B (7) **5** (1991), 341–368.
43. S. Zaidman, *Topics in Abstract Differential Equations,* Pitman Research Notes in Mathematics Ser. II John Wiley and Sons, New York, 1994–1995.
44. C. Y. Zhang, Pseudo Almost Periodic Solutions of Some Differential Equations. *J. Math. Anal. Appl* **181** (1994), no. 1, pp. 62–76.
45. C. Y. Zhang, Pseudo Almost Periodic Solutions of Some Differential Equations. II. *J. Math. Anal. Appl* **192** (1995), no. 2, pp. 543–561.
46. C. Y. Zhang, Integration of Vector-Valued Pseudo Almost Periodic Functions, *Proc. Amer. Math. Soc* **121** (1994), no. 1, pp. 167–174.

Department of Mathematics, Howard University, 2441 6th Street N.W., Washington, D.C. 20059 - USA

E-mail address: `tdiagana@howard.edu`